# 0! and Related Topics

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Most texts with anything to say about 0! usually make some such declaration as,

**0! ≡ 1 by definition.**

This has the mathematical stature of the bill passed unanimously in 1897 by the Indiana House of Representatives setting

**π ≡ 3.2**

The proper thing to say is,

**0! = 1** ***(The proof is beyond the scope of this presentation.)***

The proof ***is*** the main point, however, of this presentation. Let's begin with a modified Laplace transform called the Gamma Function [Γ(*n*)] written as

[1] $$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx \qquad n > 0$$

This is a nasty integral with no general closed form solution, but it is a valid function. Substituting *n*+1 for *n* in [1] the next order function is

[2] $$\Gamma(n+1) = \int_0^\infty e^{-x} x^{n} dx \qquad n+1 > 0$$

Integrating [2] by parts per $\int u\,dv = u \cdot v - \int v\,du$, choosing $u = x^n, v = e^{-x}dx$

[3] $$\Gamma(n+1) = -x^n e^{-x} \Big|_0^\infty + n\int_0^\infty e^{-x} x^{n-1} dx \qquad n+1 > 0$$ results.

The first term is $-\infty^n \cdot e^{-\infty} + 0^n \cdot 1 = 0$, since of $x^n$ and $e^{-x}$, the latter converges more rapidly than the former diverges. The second term is *n*·Γ(*n*) from [1], providing the recurrence relation between orders > 0 as

[4] $$\Gamma(n+1) = n\Gamma(n) \qquad n > 0$$ for the recurrence relationship.

Evaluating Γ(1) directly by setting *n* = 0 in [2], the result is

[5] $$\Gamma(1) = \Gamma(0+1) = \int_0^\infty e^{-x} dx = -e^{-x} \Big|_0^\infty = 0 - (-1) = 1$$

One obtains from the recurrence relation [4]

$$\Gamma(2) = 1 \cdot \Gamma(1) = 1 \cdot 1 = 1$$
$$\Gamma(3) = 2 \cdot \Gamma(2) = 2 \cdot 1 \cdot 1 = 2$$
$$\Gamma(4) = 3 \cdot \Gamma(3) = 3 \cdot 2 \cdot 1 \cdot 1 = 6 \quad \textit{and}$$

[6] $$\Gamma(n+1) = n \cdot \Gamma(n) = n \cdot (n-1) \cdot (n-2) \cdot \quad \cdots \quad \cdot 2 \cdot 1 \equiv n!$$

when $n$ is an integer. It is seen that the direct calculation for 0! has been done in [5] *without resort to arbitrary definition*, thus

[7] $$\Gamma(0+1) = \Gamma(1) = 0! = 1$$

**QED** = Quod Erat Demonstrandum
≠ Quite Easily Derived

Note that $0 \cdot \Gamma(0)$ need not be a concern, as the integral in [1] does not converge for $\Gamma(0) = (-1)!$, whence the $\boldsymbol{n > 0}$ stipulation. I quote MIT Professor F.B.Hildebrand on factorials of negative integers, “As it approaches negative one, it looses it’s braveness, and retreats to confusion.” Generalizing [6] as a function of a complex variable, singularities appear only at negative integer arguments, as might be expected from [4].

A rather indirect method can be used to obtain the interesting $(-\tfrac{1}{2})! = \sqrt{\pi}$ .

The very accurate Stirling approximation is given here without derivation as:

[8] $n! = \Gamma(n+1) \approx \sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n} \quad (n \to \infty)$, which is less than 8% off for $n$ as low as 1 and improves rapidly as $n$ becomes large. Accuracy is within 1% for $n \geq 9$.

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## 0! and Related Topics Continued

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Here is the rather indirect method used to obtain $(-\tfrac{1}{2})! = \sqrt{\pi}$ . [1] is rewritten in the variable $z$ as

[9] $$\Gamma(n) = \int_0^\infty e^{-z} z^{n-1} dz \qquad n > 0$$

Evaluating for $n=\tfrac{1}{2}$ [9] becomes

[10] $$\Gamma(\tfrac{1}{2}) = \int_0^\infty e^{-z} z^{-\frac{1}{2}} dz$$

Duplicating [10] and changing variables to $z=x^2$ and $z=y^2$ gives

[11] $$\Gamma(\tfrac{1}{2}) = 2\int_0^\infty e^{-x^2} dx \quad \text{and} \quad \Gamma(\tfrac{1}{2}) = 2\int_0^\infty e^{-y^2} dy$$

Multiplying the left and right sides together gives

[12] $$\left[\Gamma(½)\right]^2 = \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dx\,dy$$

This volume integral under the surface $w = e^{-(x^2+y^2)}$ is in the first quadrant x≥0,y≥0 and is converted to polar coordinates as

[13] $$\left[\Gamma(½)\right]^2 = 4\int_0^{\pi/2}\int_0^\infty e^{-r^2}\,r\,dr\,d\theta = 4\tfrac{\pi}{2}\left.\frac{e^{-r^2}}{-2}\right|_0^\infty = \pi$$

Taking the square root yields

[14] $$\Gamma(½) = (-½!) = \sqrt{\pi}$$ **QED**

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Note that [11] and [14] yield the normalization value

[15] $$\int_0^\infty e^{-x^2}dx = \tfrac{\sqrt{\pi}}{2} = \Gamma(½+1) = ½\Gamma(½)$$ from recurrence [4] applied to [14]

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A more interesting than useful equation [in these days of modern computers] given here without derivation is

[16] $$(2n)! = \tfrac{2^{2n}}{\sqrt{\pi}}\,n!(n-½)!$$